B.Sc Part II (Hons)

Homomorphism and Isomorphism

Homomorphism: — Let (G, o) and (G', o') are two groups. A mapping $f: G \to G'$ satisfying $f(aob) = f(a) o' f(b)$ $\forall$ $a, b \in G$ and $f(a), f(b)$ are their image under f then f is said to be an homomorphism of the group G with the group G'.

i.e: Let $(Z, +)$ be the additive group of integers. Let m be a fixed integer. Then the map $f: Z \to Z$ defined by $f(a) = ma$, $a \in Z$ is a homomorphism.

To prove this we suppose $a, b \in Z$ then,

$$f(a+b) = m(a+b) = ma + mb = f(a) + f(b)$$

$\therefore$ f is homomorphism.

Isomorphism: — Let (G, o) and (G', o') are two groups. A mapping $f: G \to G'$ is said to be isomorphism iff

(i) f is a homomorphism i.e. $f(a, b) = f(a) o f(b)$

(ii) f is a one-one and onto mapping

i.e. Let $G = \{ \cdots -3, -2, -, 0, 1, 2, 3, \cdots \}$ i.e. set of integers and $G' = \{ \cdots 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, \cdots \}$.

Here $(G, +)$ and $(G', .)$ are two groups. Then the function $f: G \to G'$ defined by $f(x) = 2^x$ $\forall$ $x \in G$ is and isomorphism.

Kernel of Homomorphism: Let f be a homomorphism from the group G into the group H. Let e is the identity of G and e' of H then $\ker f = \{x \in G : f(x) = e'$

Theorem: Let $f: G \to G'$ be a homomorphism of groups. Let K be the kernel of f. Then K is a normal subgroup of G.

Proof:- Let f be a homomorphism of group G into G'. Let e and e' be identities of G and G' respectively.

Let K be the kernel of f then by definition

$K = \{x \in G : f(x) = e'\}$

Ist We have to show that K is subgroup of G.

Let $a, b \in K$ then $f(a) = e'$ and $f(b) = e'$

We have $f(ab^{-1}) = f(a) f(b^{-1})$ [$\because f$ is homomorphism]

$= f(a)[f(b)]^{-1} = e'(e')^{-1} = e'e' = e'$

$\therefore$ $a^{-1}b \in K$

Hence $a, b \in K$

$\Rightarrow ab^{-1} \in K$. Therefore K is a subgroup of G.

Let $g \in G$ and x be any element of K.

Then $f(x) = e'$ [$\because K$ is kernel of f]

We have $f(gkg^{-1}) = f(g) f(k) f(g)^{-1}$

[$\because f$ is homomorphism]

$= f(g) e' (fg^{-1})$ [$\because f(k) = e'$]

$= f(g) f(g^{-1})$ [$\because e'$ is identity]

$\Rightarrow f(gkg^{-1}) = f(gg^{-1}) = f(e) = e'$

i.e. $gkg^{-1} \in K$

$\therefore K$ is a normal subgroup of a group G.

proved